

## EFFECTIVE REPRESENTATIONS OF PATH SEMIGROUPS

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ABSTRACT. We give a formula which determines the minimal effective dimensions of path semigroups and truncated path semigroups over an uncountable field of characteristic zero.

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## 1. INTRODUCTION AND PRELIMINARIES

Let  $S$  be a semigroup and  $\mathbb{k}$  a fixed field. In the paper [MS] Mazorchuk and Stienberg addressed the question of determining the so-called *minimal effective dimension*  $\text{eff.dim}_{\mathbb{k}}(S)$  of  $S$  over  $\mathbb{k}$ , that is the minimal  $m$  (a positive integer or infinity) for which there is an injective homomorphism from  $S$  to the semigroup  $\text{Mat}_{m \times m}(\mathbb{k})$  of all  $m \times m$  matrices with coefficients in  $\mathbb{k}$ . If  $S$  is finite, it is clear that  $\text{eff.dim}_{\mathbb{k}}(S) < \infty$ , more precisely,  $\text{eff.dim}_{\mathbb{k}}(S) \leq |S| + 1$ , which is the dimension of the regular representation of the semigroup  $S^1$  obtained from  $S$  by formally adjoining an identity element 1. An *effective representation* of a semigroup  $S$  with  $\text{eff.dim}_{\mathbb{k}}(S) = m$  is an injective homomorphism  $S \rightarrow \text{Mat}_{m \times m}(\mathbb{k})$ .

One of the examples considered in [MS] was that of *truncated path semigroups* which we now define. Let  $Q = (Q_0, Q_1, h, t)$  be a *quiver*, where  $Q_0$  is the set of vertices,  $Q_1$  is the set of arrows,  $h : Q_1 \rightarrow Q_0$  is the function assigning to each arrow its *head* and  $t : Q_1 \rightarrow Q_0$  is the function assigning to each arrow its *tail*. Denote by  $\mathcal{P}$  the set of all oriented paths in  $Q$  (including the trivial path  $\varepsilon_x$  at each vertex  $x \in Q_0$  and the zero path  $\mathbf{z}$ ). Then  $\mathcal{P}$  has the natural structure of a semigroup under the usual concatenation of oriented paths (in case two paths cannot be concatenated, their product is postulated to be the path  $\mathbf{z}$  and the latter is the zero element of  $\mathcal{P}$ ). The semigroup  $\mathcal{P}$  is called the *path semigroup* of  $Q$ . The semigroup  $\mathcal{P}$  is finite if and only if the quiver  $Q$  is finite and does not have oriented cycles. We write  $\mathcal{P}^*$  for the set  $\mathcal{P} \setminus \{\mathbf{z}\}$  of non-zero paths.

There is a unique function  $l : \mathcal{P}^* \rightarrow \{0, 1, 2, \dots\}$ , called the *path length*, having the properties that the length of each arrow is 1 and  $l(pq) = l(p) + l(q)$  whenever  $p, q, pq \in \mathcal{P}^*$  (note that  $l(\varepsilon_x) = 0$  for each  $x \in Q_0$ ). Elements of  $Q_1$  thus can be identified with all paths of length 1. Let  $J \subset \mathcal{P}$  be the two-sided ideal of  $\mathcal{P}$  generated by  $Q_1$ . For every  $N \in \{1, 2, 3, \dots\}$  we define the *truncated path semigroup* as  $\mathcal{P}_N := \mathcal{P}/J^N$ . Note that the semigroup  $\mathcal{P}_N$  is finite whenever  $Q$  is. In [MS, Subsection 8.1] one finds a formula for the effective dimension of  $\mathcal{P}_N$  in the case when every vertex in  $Q$  appears in some oriented cycle (or loop). The aim of this paper is give a formula for the effective dimension of  $\mathcal{P}_N$  for any  $Q$ .

From now on we assume that  $Q$  is finite and set  $n = |Q_0|$ . Consider the path algebra  $\mathbb{k}[Q]$  of  $Q$  which is the  $\mathbf{z}$ -reduced semigroup algebra of  $\mathcal{P}$  over  $\mathbb{k}$ . The algebra

$\mathbb{k}[Q]$  is unital with unit element  $1 = \sum_{x \in Q_0} \varepsilon_x$  where  $\varepsilon_x$  are pairwise orthogonal idempotents. This implies that any  $\mathbb{k}[Q]$ -module  $V$  splits as a direct sum of vector spaces

$$V = \bigoplus_{x \in Q_0} V_x,$$

where  $V_x = \varepsilon_x V$ . Given a  $\mathbb{k}[Q]$ -module  $V$ , we set  $D_x = \dim(V_x)$ . Each arrow  $\alpha : x \rightarrow y$  acts as zero on all  $V_z$  such that  $z \neq x$  and hence is uniquely determined by the induced linear map from  $V_x$  to  $V_y$ . Hence we can make the following convention: A matrix representation of  $\mathcal{P}$  or  $\mathcal{P}_N$  is an assignment to each arrow  $\alpha \in Q_1$  a  $D_y \times D_x$ -matrix with coefficients in  $\mathbb{k}$  representing the action of  $\alpha$  in fixed bases of  $V_x$  and  $V_y$ . For more details on representations of quivers we refer the reader to [GR].

Note that  $\mathcal{P}_N$ -modules are exactly  $\mathcal{P}$ -modules annihilated by  $J^N$ . We will usually denote  $\mathcal{P}$ - or  $\mathcal{P}_N$ -modules by  $V$  and the corresponding representation by  $R$ .

## 2. PATH SEMIGROUPS

**s1**

In [MS, Section 8] one finds formulae for effective dimension of path semigroups over  $\mathbb{k}$  in case of acyclic  $Q$  and algebraically closed  $\mathbb{k}$ . In this section we determine the effective dimension of path semigroups for all finite quivers at the expense of assuming  $\mathbb{k}$  to be a field containing an infinite purely transcendental extension of its prime subfield (for example,  $\mathbb{R} \subset \mathbb{k}$ ). We denote by  $\mathbb{N}$  the set of positive integers and by  $\mathbb{N}_0$  the set of non-negative integers.

For  $x \in Q_0$  let  $\mathcal{P}_x$  denote the set of all paths in  $\mathcal{P}^*$  which start and terminate at  $x$ . Then  $\mathcal{P}_x$  is a subsemigroup of  $\mathcal{P}$ , in fact,  $\mathcal{P}_x$  is a monoid with identity  $\varepsilon_x$ . Denote by  $A$  the set of all vertices  $x \in Q_0$  for which  $\mathcal{P}_x$  is not commutative and set  $B := Q_0 \setminus A$ .

**lem101**

**Lemma 1.** *Let  $x \in Q_0$ .*

**lem101.1**

(i) *The monoid  $\mathcal{P}_x$  has a unique irreducible generating system (which we denote by  $M_x$ ).*

**lem101.2**

(ii) *The monoid  $\mathcal{P}_x$  is free over  $M_x$ .*

**lem101.3**

(iii) *The monoid  $\mathcal{P}_x$  is commutative if and only if  $|M_x| \leq 1$ .*

*Proof.* Define  $N_i$  and  $\tilde{N}_i$  for  $i \in \mathbb{N}$  recursively as follows:

- $N_1$  is the set of all paths of length 1 in  $\mathcal{P}_x$ ;
- $\tilde{N}_1$  is the subsemigroup of  $\mathcal{P}_x$  generated by  $N_1$ ;
- $N_2$  is the set of all paths of length 2 in  $\mathcal{P}_x \setminus \tilde{N}_1$ ;
- $\tilde{N}_2$  is the subsemigroup of  $\mathcal{P}_x$  generated by  $N_1 \cup N_2$ ;
- $N_3$  is the set of all paths of length 3 in  $\mathcal{P}_x \setminus \tilde{N}_2$ ;
- $\tilde{N}_3$  is the subsemigroup of  $\mathcal{P}_x$  generated by  $N_1 \cup N_2 \cup N_3$ ;
- and so on.

From this definition it is clear that the set  $M_x = N_1 \cup N_2 \cup \dots$  is a generating system of  $\mathcal{P}_x$  (as a monoid) and that it is included in every generating system of  $\mathcal{P}_x$  (as a monoid). Claim (i) follows.

Assume that  $\mathcal{P}_x$  is not free over  $M_x$ . Then there exist  $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l \in M_x$  such that  $a_1 a_2 \dots a_k = b_1 b_2 \dots b_l$ . This can be chosen such that  $(k, l)$  is minimal possible with respect to the lexicographic order (note that, obviously, both  $k, l > 0$ ). If  $l(a_k) < l(b_l)$ , then  $b_l = t a_k$  for some  $t \in \mathcal{P}_x$  which contradicts  $b_l \in M_x$ . Therefore this case is not possible. Similarly  $l(a_k) > l(b_l)$  is not possible. This means that  $l(a_k) = l(b_l)$  and hence  $a_k = b_l$ . Therefore  $a_1 a_2 \dots a_{k-1} = b_1 b_2 \dots b_{l-1}$  which contradicts minimality of  $(k, l)$ . This proves claim (ii) and claim (iii) follows directly from claim (ii).  $\square$

A generator of  $\mathcal{P}_x$  will be called a *minimal oriented cycle* starting at  $x$

**lem102**

**Lemma 2.** *Let  $V$  be an effective  $S$ -module and  $x \in A$ . Then  $D_x \geq 2$  and*

$$\text{eff.dim}_{\mathbb{K}}(S) \geq 2|A| + |B| = |A| + n.$$

*Proof.* It is clear that  $D_x \geq 1$  for all  $x \in Q_0$  (for otherwise the actions of  $\varepsilon_x$  and  $\mathbf{z}$  would coincide). Assume  $x \in A$  and  $D_x = 1$ . Then  $\mathcal{P}_x$  acts effectively on the 1-dimensional vector space  $V_x$ . However, the semigroup of linear endomorphisms of  $V_x$  is commutative (as  $V_x$  is one dimensional), while  $\mathcal{P}_x$  is not (as  $x \in A$ ), a contradiction. This implies that  $D_x > 1$  for  $x \in A$ , that is  $D_x \geq 2$ . As  $|A| + |B| = |Q_0| = n$ , the claim of the lemma follows.  $\square$

To prove that the bound given by Lemma 2 is sharp, we will need the following construction: For a fixed positive integer  $k$  consider the alphabet  $A = \{a_1, a_2, \dots, a_k\}$  and the free monoid  $A^*$  of all finite words over  $A$  with respect to concatenation of words. Let  $\mathbb{K}_k$  be the purely transcendental extension of its prime subfield  $\mathbb{K}$  with basis  $\mathbf{B} := \{\tau_i, \eta_i, \zeta_i \mid i = 1, 2, \dots, k\}$ .

**lem105**

**Lemma 3.** *There is a unique representation  $R : A^* \rightarrow \text{Mat}_{2 \times 2}(\mathbb{K}_k)$  such that*

$$R(a_i) = \begin{pmatrix} \tau_i & \eta_i \\ 0 & \zeta_i \end{pmatrix},$$

*moreover, the map  $R$  is injective.*

*Proof.* Existence and uniqueness of  $R$  follows from the fact that  $A^*$  is free over  $A$ . For  $a_{i_1} a_{i_2} \dots a_{i_l} \in A^*$  the coefficient in the first row and second column of the matrix  $R(a_{i_1} a_{i_2} \dots a_{i_l})$  equals

$$\sum_{i=1}^l \tau_1 \tau_2 \dots \tau_{i-1} \eta_i \zeta_{i+1} \zeta_{i+2} \dots \zeta_l.$$

This uniquely determines the sequence  $i_1, i_2, \dots, i_l$  and the claim about injectivity follows.  $\square$

For a fixed  $Q$  let  $\mathbb{K}_Q$  be the purely transcendental extension of its prime subfield  $\mathbb{K}$  with basis  $\mathbf{B} := \{\tau_\alpha, \eta_\alpha, \zeta_\alpha \mid \alpha \in Q_1\}$ . For  $\alpha \in Q_1$  set  $\mathbf{B}_\alpha := \{\tau_\alpha, \eta_\alpha, \zeta_\alpha\}$ . For  $x, y \in Q_0$  write  $x \sim y$  if  $x = y$  or there is an oriented path from  $x$  to  $y$  as well as an oriented path from  $y$  to  $x$ .

**lem107** **Lemma 4.** *Let  $x, y \in Q_0$  be such that  $x \sim y$ . Then  $x \in A$  if and only if  $y \in A$ .*

*Proof.* As  $x \sim y$ , there exist paths  $\omega_{xy} : x \rightarrow y$  and  $\omega_{yx} : y \rightarrow x$ . Assume  $x \in A$ . Let  $\omega_1$  and  $\omega_2$  be two different minimal oriented cycles in  $\mathcal{P}_x$ . Then  $\omega_{xy}\omega_1\omega_{yx}$  and  $\omega_{xy}\omega_2\omega_{yx}$  are two noncommuting elements in  $\mathcal{P}_x$ , proving  $y \in A$ . Claim now follows by symmetry.  $\square$

The following is our first main result.

**thm21** **Theorem 5.** *Let  $Q$  be a finite quiver and  $\mathcal{P}$  the corresponding path semigroup. Then*

$$\text{eff.dim}_{\mathbb{k}_Q}(\mathcal{P}) = |A| + n.$$

*Proof.* We only need to show that the bound given by Lemma 2 is sharp. To do this we construct an effective matrix representation  $V$  of  $\mathcal{P}$  as follows: set

$$D_x = \dim(V_x) = \begin{cases} 2, & x \in A; \\ 1, & x \in B; \end{cases}$$

with a fixed basis in each  $V_x$ . To each  $\alpha \in Q_1$  we assign a  $\mathbb{k}_Q$ -matrix with  $D_{h(\alpha)}$  rows and  $D_{t(\alpha)}$  columns by the following rule:

- If  $h(\alpha_i), t(\alpha_i) \in A$ , then we assign to  $\alpha$  the matrix  $\begin{pmatrix} \tau_\alpha & \eta_\alpha \\ 0 & \zeta_\alpha \end{pmatrix}$ .
- If  $h(\alpha_i) \in A$  and  $t(\alpha_i) \in B$ , then we assign to  $\alpha$  the matrix  $(\tau_\alpha \ \zeta_\alpha)$ .
- If  $h(\alpha_i) \in B$  and  $t(\alpha_i) \in A$ , then we assign to  $\alpha$  the matrix  $\begin{pmatrix} \tau_\alpha \\ \zeta_\alpha \end{pmatrix}$ .
- If  $h(\alpha_i), t(\alpha_i) \in B$ , then we assign to  $\alpha$  the matrix  $(\tau_\alpha)$ .

Finally, to each  $\varepsilon_x$  we assign the identity matrix of size  $D_x$  and to each path of length more than 1 the corresponding product of the matrices assigned to arrows which this path consists of. It is obvious that this gives a well-defined representation of  $\mathcal{P}$ . It remains to show that this representation sends different elements of  $\mathcal{P}$  to different linear operators.

Let  $x, y \in Q_0$  and  $\omega$  be an oriented paths from  $x$  to  $y$ . Directly from the above construction it follows that each coefficient of the matrix representing  $\omega$  is a homogeneous polynomial in elements from  $\mathbf{B}$ . If this coefficient is nonzero (which is the case for all diagonal entries and all entries above the diagonal, in case the latter exist), this polynomial has degree  $l(\omega)$  and depends on at least one element from  $\{\tau_\alpha, \eta_\alpha, \zeta_\alpha\}$  for each arrow  $\alpha$  in  $\omega$ .

Let  $x, y \in Q_0$  and  $\omega, \omega'$  be two paths from  $x$  to  $y$ . We have to show that  $\omega$  and  $\eta$  are represented by different linear operators. From the previous paragraph it follows that this is clear in the case when  $\omega$  and  $\omega'$  have different lengths and in the case when one of these paths contains an arrow which is not contained in the other path.

Assume that there exists  $x, y \in Q_0$  and  $\omega, \omega'$  two different paths from  $x$  to  $y$  such that  $R(\omega) = R(\omega')$ . Without loss of generality we may assume that the pair  $(l(x), l(y))$  is minimal with respect to the lexicographic order.

Write  $\omega$  in the form  $\omega_1\beta_1\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k$  where  $\omega_i$  are (possibly trivial) paths inside an equivalence class of the relation  $\sim$  and  $\beta_i$  are arrow between equivalence classes. From the above it then follows that  $\omega'$  can similarly be written as  $\omega'_1\beta_1\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k$ .

Assume  $\omega_1$  is a trivial path. Then  $\omega$  has no arrow starting from the  $\sim$ -equivalence class of  $h(\beta_1)$ . From the above we get that  $\omega'$  has no arrow starting from the  $\sim$ -equivalence class of  $h(\beta_1)$  and hence  $\omega'_1$  is a trivial path as well. We claim that this implies

$$\boxed{\text{eq1}} \quad (2.1) \quad R(\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k) = R(\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k)$$

which would then contradict the minimality of  $(l(x), l(y))$ . To prove (2.1), the only non-trivial case to consider is when  $R(\beta_1)$  is not injective, that is  $t(\beta_1) \in A$  and  $h(\beta_1) \in B$ . Assume

$$R(\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \neq \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = R(\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k).$$

Then none of  $a, b, c, a', b', c'$  depends on  $\tau_{\beta_1}$  or  $\zeta_{\beta_1}$  and hence we have

$$\begin{aligned} R(\omega) &= \begin{pmatrix} \tau_{\beta_1} & \zeta_{\beta_1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \tau_{\beta_1}a & \tau_{\beta_1}b + \zeta_{\beta_1}c \end{pmatrix} \neq \\ &\neq \begin{pmatrix} \tau_{\beta_1}a' & \tau_{\beta_1}b' + \zeta_{\beta_1}c' \end{pmatrix} = \begin{pmatrix} \tau_{\beta_1} & \zeta_{\beta_1} \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = R(\omega'), \end{aligned}$$

a contradiction.

Therefore  $\omega_1$  is non-trivial and thus  $R(\omega_1)$  is invertible by construction and Lemma 4 as both the starting point and the ending point of  $\omega_1$  belong to the same  $\sim$ -equivalence class. Multiplying with  $R(\omega_1)^{-1}$  we get

$$R(\beta_1\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k) = R(\omega_1)^{-1}R(\omega'_1)R(\beta_1\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k).$$

Note that the left hand side does not depend on elements in  $\mathbf{B}_\alpha$  for  $\alpha$  occurring in  $\omega_1$ . Hence the right hand side does not depend on these elements either which forces the injective linear map  $R(\omega_1)^{-1}R(\omega'_1)$  to be the identity linear map as the image of the linear map  $R(\beta_1\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k)$  is nonzero by construction. Therefore in this case we have the equality  $R(\omega_1) = R(\omega'_1)$ . If  $\beta_1\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k$  or  $\beta_1\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k$  is non-trivial, the above gives

$$R(\beta_1\omega_2\beta_2\cdots\omega_{k-1}\beta_{k-1}\omega_k) = R(\beta_1\omega'_2\beta_2\cdots\omega'_{k-1}\beta_{k-1}\omega'_k)$$

which contradicts minimality of  $(l(x), l(y))$ . Hence  $\omega = \omega_1$  and  $\omega' = \omega'_1$ .

If  $x \in A$ , then  $R(\omega) = R(\omega')$  implies  $\omega = \omega'$  by Lemma 3, a contradiction. Therefore  $x, y \in B$ . In this case there is a unique minimal oriented cycle  $q$  from  $x$  to  $x$  ( $q$  may be a trivial path) and hence a unique path  $p$  of minimal length from  $x$  to  $y$  (for otherwise, composing two different such minimal paths from  $x$  to  $y$  with a minimal path from  $y$  to  $x$  we would get two minimal oriented cycles from  $x$  to  $x$ ). Any path from  $x$  to  $y$  has thus the form  $pq^l$  for some positive integer  $l$ . In particular, two paths of the same length from  $x$  to  $y$  must coincide, which contradicts our choice of  $\omega$  and  $\omega'$ . This final contradiction completes the proof of the theorem.  $\square$

s3

## 3. TRUNCATED PATH SEMIGROUPS

As truncated path semigroups are obtained by adding some relations to usual path semigroups, it is reasonable to expect that the effective dimension increases, e.g. compare the statements of Theorem 5 above with the results of [MS, Subsection 8.2].

Let  $\mathbb{k}$  be any field,  $N \in \mathbb{N}$  and  $V$  a representation of  $\mathcal{P}_N$ . For every  $k \in \mathbb{N}_0$  let  $W^{(k)} = \text{span}\{\omega V \mid \omega \in \mathcal{P}, l(\omega) = k\}$ . By convention,  $\omega = \mathbf{z}$  when  $l(\omega) \geq N$ , which gives  $W^{(N)} = 0$ . Thus we get the chain of subspaces

$$V = W^{(0)} \supset W^{(1)} \supset \dots \supset W^{(N-1)} \supset W^{(N)} = 0.$$

For every  $x \in Q_0$  set  $W_x^{(k)} := V_x \cap W^{(k)}$  and choose *some*  $V_x^{(k)} \subset W_x^{(k)}$  such that  $W_x^{(k)} = V_x^{(k)} \oplus W_x^{(k+1)}$ . Set  $V^{(k)} := \bigoplus_{x \in Q_0} V_x^{(k)}$ . This gives the vector space decompositions

$$V = \bigoplus_{i=0}^{N-1} V^{(i)} = \bigoplus_{x \in Q_0} V_x = \bigoplus_{\substack{0 \leq i \leq N-1 \\ x \in Q_0}} V_x^{(i)}.$$

In any module  $V$ , let  $D_x^{(i)} := \dim(V_x^{(i)})$ , which gives  $D_x = \sum_{i=0}^{N-1} D_x^{(i)}$ . From the definition of  $W^{(i)}$  for any  $\omega \in \mathcal{P}$  we have  $\omega W^{(i)} \subset W_{h(\omega)}^{(i+l(\omega))}$ .

lem202

**Lemma 6.** *Let  $x \in Q_0$  be such that there are paths  $\omega_l, \omega_r$  and  $0 \leq k < N$  such that  $l(\omega_l) = k$ ,  $l(\omega_r) = N - 1 - k$  and  $h(\omega_l) = t(\omega_r) = x$ . Then  $D_x^{(k)} \geq 1$  for every effective  $\mathcal{P}$ -module  $V$ .*

*Proof.* Assume  $D_x^{(k)} = 0$ , that is  $V_x^{(k)} = 0$ , and let  $y = t(\omega_l)$  and  $z = h(\omega_r)$ . Then

$$\omega_l(V) = \omega_l(V_y) = \omega_l(W_y^{(0)}) \subset W_x^{(l(\omega_l))} = W_x^{(k)} = V_x^{(k)} \oplus W_x^{(k+1)} = W_x^{(k+1)} \text{ and}$$

$$\omega_r(W_x^{(k+1)}) \subset W_z^{(k+1+l(\omega_r))} = W_z^{(k+1+N-1-k)} = W_z^{(N)} = 0.$$

Thus  $\omega_r \omega_l(V) = 0$  and  $\omega_r \omega_l$  acts as  $\mathbf{z}$  on  $V$  contradicting effectiveness.  $\square$

For  $x \in Q_0$  define

$$K(x) := \{k \in \{0, \dots, N-1\} \mid \text{there are paths } \omega_l, \omega_r \text{ such that } l(\omega_l) = k, l(\omega_r) = N-1-k \text{ and } h(\omega_l) = t(\omega_r) = x\}.$$

Set  $B := \{x \in Q_0 \mid K(x) = \emptyset\}$  and  $A := Q_0 \setminus B$ . For  $x \in A$  set

$$\underline{k}_x := \min(K(x)) \quad \text{and} \quad \overline{k}_x := \max(K(x)).$$

For  $x \in Q_0$  define

$$l_x^- := \sup\{l(\omega) \mid \omega \in \mathcal{P} \text{ and } h(\omega) = x\} \text{ and } l_x^+ := \sup\{l(\omega) \mid \omega \in \mathcal{P} \text{ and } t(\omega) = x\}.$$

We are now ready to state our second main result.

thm205

**Theorem 7.** *Define  $d_x := \min\{l_x^- + 1, l_x^+ + 1, N, \max\{l_x^- + l_x^+ + 2 - N, 1\}\}$ .*

thm205.1

(i) *For every effective  $\mathcal{P}_N$ -module  $V$  over any field  $\mathbb{k}$  we have  $D_x \geq d_x$ .*

thm205.2

(ii) *If  $\mathbb{k}$  has characteristic zero or is uncountable, then  $D_x = d_x$  for some effective  $\mathcal{P}_N$ -module (over  $\mathbb{k}$ ) and  $\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_N) = \sum_{x \in Q_0} d_x$ .*

*Proof.* First we **prove claim** (i). Let  $x \in Q_0$ . Then  $x \in A$  or  $x \in B$ . In any case,  $D_x \geq |K(x)|$  by Lemma 6.

**Assume first that**  $x \in A$ . Then  $K(x) \neq \emptyset$  and it suffices to show that  $|K(x)| \geq d_x$ . As  $K(x) \neq \emptyset$ , there is some path of length  $N - 1$  passing through  $x$ , which means that  $l_x^- + l_x^+ \geq N - 1$ , in particular,  $l_x^- + l_x^+ - N + 2 \geq 1$  and thus  $\max\{l_x^- + l_x^+ + 2 - N, 1\} = l_x^- + l_x^+ - N + 2$ .

Pick some paths  $\omega_-, \omega_+$  such that  $h(\omega_-) = t(\omega_+) = x$  and  $l(\omega_\pm) = \min(l_x^\pm, N - 1)$ . Let  $\omega_{\min(l_x^-, N-1)}$  be a path of length  $N - 1$  that starts with  $\omega_-$  and continues into  $\omega_+$  (if needed). From Lemma 6 we get  $\min(l_x^-, N - 1) \in K(x)$ . Now we repeat recursively the following procedure as long as possible: Change  $\omega_k$  to  $\omega_{k-1}$  by removing the tail arrow and adding a new head arrow from  $\omega_+$ . On each step of this procedure we get a new  $\omega_{k-1}$  with  $k - 1 \in K(x)$ . This procedure can stop for two reasons:

- There are no more arrows from  $\omega_-$  to remove.
- There are no more arrows from  $\omega_+$  to add.

The first case (there are no more arrows from  $\omega_-$  to remove) can only happen if the latest  $k - 1$  created is equal to 0. In this case  $K(x) \supset \{0, 1, \dots, \min(l_x^-, N - 1)\}$  and hence  $|K(x)| \geq \min(l_x^- + 1, N)$  and we are done.

We split the second case (there are no more arrows from  $\omega_+$  to add) into two subcases. The first subcase is that  $\omega_{\min(l_x^-, N-1)} = \omega_-$ , that is  $l_x^- \geq N - 1$ . In this subcase we have  $K(x) \supset \{N - 1, N - 2, \dots, N - 1 - \min(l_x^+, N - 1)\}$  which implies that  $|K(x)| \geq \min(l_x^+ + 1, N)$  and we are done.

The second subcase is when  $\omega_{\min(l_x^-, N-1)} \neq \omega_-$ . In this subcase we have  $l_x^- < N - 1$  and

$$K(x) \supset T := \{l_x^-, l_x^- - 1, \dots, N - 1 - \min(l_x^+, N - 1)\}.$$

Hence  $|K(x)| \geq |T| = (l_x^- + \min(l_x^+, N - 1) + 2 - N)$ . If  $\min(l_x^+, N - 1) = l_x^+$ , this gives  $|K(x)| \geq l_x^- + l_x^+ + 2 - N$  and we are done. If  $\min(l_x^+, N - 1) = N - 1$ , this gives  $|K(x)| \geq l_x^- + 1$  and we are done. This completes verification of  $D_x \geq |K(x)| \geq d_x$  for  $x \in A$ .

**Assume now that**  $x \in B$ . In this case  $l_x^- + l_x^+ + 2 - N \leq 0$  and  $d_x = 1$ . The fact that  $D_x \geq 1$  is clear as  $\varepsilon_x$  acts as the identity on  $V_x$  and this should be different from the action of  $\mathbf{z}$  which acts as zero. This completes the proof of claim (i) and implies

$$\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_N) \geq \sum_{x \in Q_0} d_x.$$

To **prove claim** (ii) we assume that  $\mathbb{k}$  has characteristic zero or is uncountable. We have to construct an effective representation  $V$  such that  $D_x = d_x$  for every  $x \in Q_0$ . To do this we define the following:

- for  $x \in A$  and  $k \in K(x)$  let  $V_x^{(k)}$  be the one-dimensional vector space with basis  $\{v_x^{(k)}\}$ ;
- for  $x \in A$  and  $k \notin K(x)$  let  $V_x^{(k)}$  be the zero vector space;

- for  $x \in B$  let  $V_x$  be the one-dimensional vector space with basis  $\{v_x\}$ .

Set

$$V := \left( \bigoplus_{\substack{0 \leq i \leq N-1 \\ x \in A}} V_x^{(i)} \right) \oplus \left( \bigoplus_{x \in B} V_x \right).$$

Fix an injective map  $(\alpha, k) \mapsto p_{\alpha, k}$  from the set of all pairs  $(\alpha, k)$  where  $\alpha \in Q_1$  and  $0 \leq k \leq N$  to the set of positive integer prime numbers if  $\mathbb{k}$  has characteristic 0. In case  $\mathbb{k}$  is uncountable we choose the codomain as a basis of a purely transcendental extension over its prime subfield by sufficiently many base elements. Define the action of  $\mathcal{P}_N$  on  $V$  as follows:

- the zero element of  $\mathcal{P}_N$  acts as zero;
- $\varepsilon_x$  acts as the identity on  $V_x$  and as zero on  $V_y$ ,  $y \neq x$ ;
- for every arrow  $\alpha : x \rightarrow y$  with  $x, y \in A$  we have  $\alpha : v_x^{(N-1)} \rightarrow 0$  and for each  $k \in K(x)$  we have  $\alpha : v_x^{(k)} \rightarrow p_{\alpha, k} v_y^{(j)}$ , where
$$j = \min\{i \in \{k+1, k+2, \dots, N-1\} \mid V_y^{(i)} \neq 0\};$$
- for every arrow  $\alpha : x \rightarrow y$  with  $x \in A$  and  $y \in B$  and for each  $k \in K(x)$  we have  $\alpha : v_x^{(k)} \rightarrow p_{\alpha, k} v_y$ ;
- for every arrow  $\alpha : x \rightarrow y$  with  $x \in B$  and  $y \in A$  we have  $\alpha : v_x \rightarrow p_{\alpha, 0} v_y^{(\underline{k}_y)}$ ;
- for every arrow  $\alpha : x \rightarrow y$  with  $x, y \in B$  we have  $\alpha : v_x \rightarrow p_{\alpha, 0} v_y$ ;
- actions of paths of length greater than one are defined using composition of maps.

Assume that  $x, y \in A$  and  $k \in K(x)$ . Let  $\omega_-$  and  $\omega_+$  be two paths such that  $l(\omega_-) = k$ ,  $l(\omega_+) = N-1-k$  and  $h(\omega_-) = t(\omega_+) = x$ . Assume further that there is an arrow  $\alpha$  from  $x$  to  $y$ . If  $N-1-k \leq l_y^+ + 1$ , then without loss of generality we may assume that  $\alpha$  is the first arrow in  $\omega_+$ . In this case we directly get  $k+1 \in K(y)$ . If  $l_y^+ + 1 < N-1-k$ , then any  $k' \in K(y)$  satisfies  $N-1-k' \leq l_y^+ < N-1-k$  which implies  $k' > k$ . Since  $K(y)$  is not empty (as  $y \in A$ ), we get that the set  $\{i \in \{k+1, k+2, \dots, N-1\} \mid V_y^{(i)} \neq 0\}$  is non-empty. Therefore the above definitions make sense.

The only non-trivial relation to check is the fact that any path  $\omega$  with  $l(\omega) \geq N$  acts as zero. From the definition of  $B$  it follows that each arrow in  $\omega$  is an arrow between two vartices in  $A$ . From the definition of the action we then see that

$$\omega(V_{t(\omega)}^{(i)}) \subset V_{h(\omega)}^{(i+l(\omega))}.$$

This implies  $\omega(V) \subset 0$  and thus  $V$  is a  $\mathcal{P}_N$ -module.

It remains to show that our module is effective. For this we need to show that paths of length at most  $N-1$  act in a non-zero way and pairwise differently. A path  $\omega$  is said to be *maximal* if there is no arrow  $\alpha$  such that  $\alpha\omega$  or  $\omega\alpha$  is nonzero. Note that if a path  $\omega$  acts in a nonzero way, then  $h(\omega)$  can be recovered as the unique  $y$  such that  $\omega(V) \subset V_y$ . Moreover,  $t(\omega)$  can be recovered as the unique  $x$  such that  $\omega(V_x) \neq 0$ . Thus if two different paths  $\omega_1$  and  $\omega_2$  act equally and in a nonzero way, then they share the same head and the same tail. Furthermore, the action of each



maximal path  $\omega_l \omega_1 \omega_r$  coincides with the action of  $\omega_l \omega_2 \omega_r$ . Thus it suffices to show that all maximal paths act nonzero and differently.

To simplify notation let

$$\hat{v}_x := \begin{cases} v_x^{(\underline{k}_x)}, & x \in A; \\ v_x, & x \in B; \end{cases} \quad \check{v}_y := \begin{cases} v_y^{(\overline{k}_y)}, & y \in A; \\ v_y, & y \in B. \end{cases}$$

Let  $\omega = \alpha_{N-1} \alpha_{N-2} \cdots \alpha_2 \alpha_1$  be a path of length  $N-1$  and set  $x_i := h(\alpha_i) = t(\alpha_{i+1})$  with  $x_0 := t(\alpha_1)$ . Then from Lemma 6 and our construction we get that  $V_{x_i}^{(i)}$  is nonzero for all  $i$  and  $\omega(\hat{v}_x) = p_{\alpha_1,0} p_{\alpha_2,1} \cdots p_{\alpha_{N-1},N-2} \check{v}_y$ . Injectivity of the map  $(\alpha, k) \mapsto p_{\alpha,k}$  guarantees that the coefficient at  $\check{v}_y$  uniquely determines the sequence  $(\alpha_1, 0), (\alpha_2, 1), \dots, (\alpha_{N-1}, N-2)$  which uniquely determines  $\omega$ .

Finally, assume that  $\omega = \alpha_k \alpha_{k-1} \cdots \alpha_2 \alpha_1$  for some  $k < N-1$ . Set  $w_0 = \hat{v}_x$  and  $w_i = \alpha_i \cdots \alpha_2 \alpha_1(\hat{v}_x)$  for  $i = 1, 2, \dots, k$ . Let us prove that  $w_i$  is nonzero for all  $i = 0, 1, 2, \dots, k$  by induction. The basis is obvious. Assume  $w_i \neq 0$ . If  $\alpha_{i+1}$  is adjacent to at least one vertex in  $B$ , we have  $\alpha_{i+1}(w_i) \neq 0$  directly by construction. Assume now that  $\alpha_{i+1}$  is an arrow between two vertices in  $A$ . By construction, the only basis element in  $V_{t(\alpha_{i+1})}$  which  $\alpha_{i+1}$  annihilates is the one which is in the image of some path of length  $N-2$ . We have  $i < N-2$ . Hence  $\alpha_{i+1} w_i \neq 0$  if  $t(\alpha_j) \in A$  for all  $j \leq i$ . Otherwise let  $j$  be maximal such that  $j \leq i$  and  $t(\alpha_j) \in B$ . Then, by construction,  $\alpha_j(w_{j-1})$  is a non-zero multiple of  $\hat{v}_{h(\alpha_j)}$ , which implies that  $w_i$  is not in the image of a path of length  $N-2$  and therefore  $\alpha_{i+1}(w_i) \neq 0$  again. This shows that  $\omega$  acts in a nonzero way on  $V$ . As  $\omega$  is a maximal path of length strictly less than  $N-1$ , it is uniquely determined by the arrows it consists of. Injectivity of the map  $(\alpha, k) \mapsto p_{\alpha,k}$  thus implies that  $\omega$  is uniquely determined by the prime decomposition of the coefficients in its matrix. This completes the proof.  $\square$

Theorem 7 implies the following stabilization property for  $\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_N)$ :

**cor217**

**Corollary 8.** *Assume that  $\mathbb{k}$  has characteristic zero. Then there exist  $a, b \in \mathbb{N}_0$  such that*

$$\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_N) = aN + b \quad \text{for all } N \geq n.$$

*Proof.* For each  $x$  the numbers  $l_x^-$  and  $l_x^+$  satisfy  $l_x^- + l_x^+ \in \{0, 1, \dots, n-1, \infty\}$  as any path of length at least  $n$  must contain a subcycle. This means that we always have one of the following three cases:

- Both  $l_x^-$  and  $l_x^+$  are finite, and thus  $l_x^- + l_x^+ \leq n-1$ . Then for all  $N > n$  we have  $l_x^- + l_x^+ + 2 - N \leq 1$  and  $d_x = 1$ .
- Exactly one of  $l_x^-, l_x^+$  is finite. Then  $d_x = \min\{l_x^-, l_x^+\} + 1$  for all  $N \geq n$ .
- Both  $l_x^-, l_x^+$  are infinite. Then  $d_x = N$  for all  $N \geq 1$ .

Therefore we can take  $a$  to be the number of  $x$  such that both  $l_x^-, l_x^+$  are infinite. As  $b$  we take the sum of 1's over all  $x$  such that both  $l_x^-$  and  $l_x^+$  are finite plus the sum of  $\min\{l_x^-, l_x^+\} + 1$  over all  $x$  such that exactly one of  $l_x^-$  and  $l_x^+$  is finite. The claim follows.  $\square$

From Corollary 8 it follows that to calculate  $\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_N)$  for all  $N \in \mathbb{N}$  it is enough to consider the cases  $N = 1, 2, \dots, n, n+1$ .

s4s4

## 4. EXAMPLES

**4.1. Quivers with cycles at each vertex.** Let  $Q$  be a quiver in which every vertex is part of some (nontrivial) cycle or loop. Then  $\text{eff.dim}_{\mathbb{k}}(\mathcal{P}_{\mathcal{N}}) = Nn$  for  $\mathbb{k}$  uncountable or of characteristic 0. Proof: Let  $x \in Q_0$ . Then  $l_x^- = l_x^+ = \infty$  and hence  $d_x = N$ . Sum over all vertices. This result is similar to [MS, Theorem 31], but the set of fields  $\mathbb{k}$  differ.

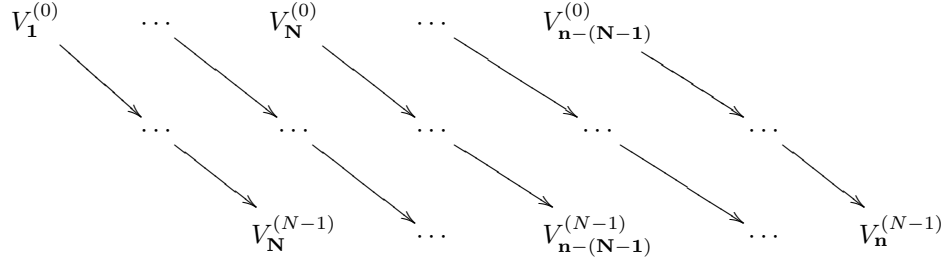
s4.2

**4.2. Quivers of type  $A_n$ .** A quiver  $Q$  is said to be of type  $A_n$  if the underlying unoriented graph is the Dynkin diagram  $A_n$ . Let  $Q$  be of type  $A_n$  and let  $(n_1, n_2, \dots, n_k)$  be the the number of vertices in the ordered segments. Then

$$\text{eff.dim}_{\mathbb{k}}(\mathcal{P}) = 1 + \sum_{N < n_i} (N(n_i + 1 - N) - 1) + \sum_{n_i \leq N} (n_i - 1).$$

Proof: Because local dimensions  $d_x$  only depend on maximal paths in and out of  $x$ , different ordered segments can be counted independently, if we subtract the overlaps. Thus we need only to consider the case when  $Q$  has one ordered segment. For a quiver of type  $A_n$  with only one ordered segment (with vertices from  $\mathbf{1}$  to  $\mathbf{n}$ ) the picture is as follows, when  $N < n$ . When  $n \leq N$  each  $V_x$  is one-dimensional.

$$\mathbf{1} \longrightarrow \dots \longrightarrow \mathbf{N} \longrightarrow \dots \longrightarrow \mathbf{n} - (\mathbf{N} - \mathbf{1}) \longrightarrow \dots \longrightarrow \mathbf{n}$$



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